

Problem Set 11 – Statistical Physics B

Problem 1: Free energy minimisation

In the lecture, we discussed that for an isolated system the internal energy is minimal. Show that for a system that can only exchange energy with its surroundings, that the Helmholtz free energy is minimal in equilibrium.

Problem 2: Heat equation

- (a) Empirically it is known that the heat or energy current density is related to temperature gradients via the law of Fourier $\mathbf{J}_\epsilon = -k\nabla T$, with k the thermal conductivity. Connect the kinetic coefficient $L_{\epsilon\epsilon}$ to k .

- (b) The heat equation is

$$\frac{\partial}{\partial t}T(\mathbf{r}, t) = \kappa\nabla^2T(\mathbf{r}, t),$$

with κ the thermal diffusivity. Express κ in terms of the number density ρ , isobaric heat capacity per particle c_P , and $L_{\epsilon\epsilon}$. You may assume that the material is a solid or liquid.

- (c) Consider now a system that is translationally invariant in the (x, y) plane, i.e. $T(\mathbf{r}, t) = T(z, t)$. We impose boundary conditions $\lim_{z \rightarrow \pm\infty} T(z, t) = T_\pm, \forall t > 0$. As the initial condition, we take $T(z, 0) = T_- + \Theta(z)(T_+ - T_-)$. Determine $T(z, t)$. In particular show that $T(z, t) = T(z/\xi)$, and give an expression for ξ . How do you interpret this result?

- (d) Show that for $|z/\xi| \ll 1$, we find

$$T(z, t) \sim \frac{1}{2}(T_+ + T_-) + \frac{T_+ - T_-}{\sqrt{\pi}\xi(t)}z,$$

and give a physical interpretation for $\xi(t)$.

- (e) Determine the entropy production density $\sigma(\mathbf{r}, t)$ and plot this quantity for several values of $\xi(t)$. What do you conclude? How does the total entropy production $\int d\mathbf{r} \sigma$ evolve as a function of t ?

Problem 3: Entropy fluctuations

- (a) Use the Einstein fluctuation theory to prove that

$$\left\langle \frac{\partial \Delta S}{\partial \alpha_i} \alpha_j \right\rangle = -k_B \delta_{ij},$$

where $\Delta S = S(E, \boldsymbol{\rho}) - S(E, \boldsymbol{\rho}_{\text{eq}})$ and $\boldsymbol{\alpha} = \boldsymbol{\rho} - \boldsymbol{\rho}_{\text{eq}}$, with $\boldsymbol{\rho}$ the generalised density vector. What are examples of components of $\boldsymbol{\rho}$?

- (b) Use the result of the previous question to show that

$$\langle \boldsymbol{\alpha} \boldsymbol{\alpha} \rangle = k_B \mathbf{g}^{-1},$$

with $g_{ij} = \partial^2 S / \partial \rho_i \partial \rho_j |_{\boldsymbol{\rho}=\boldsymbol{\rho}_{\text{eq}}}$.

Problem 4: Principle of minimum entropy production

The principle of minimum entropy production shows that under certain circumstances the total entropy production is minimised for a stationary state, where the state of the system does not evolve in time.

- (a) Is a stationary state necessarily an equilibrium state? If not, what is the difference?
- (b) Consider a one-component solid with some fixed temperature distribution on the walls in which the system is enclosed. Write down the equation for the temperature distribution assuming that the kinetic coefficient $L_{\epsilon\epsilon}$ is a constant. What is the equation for the stationary state?
- (c) Write down an expression for $P = \int_V d\mathbf{r} \sigma$. What temperature distributions minimize this functional?
- (d) Prove that

$$\frac{\partial P}{\partial t} = -2 \int_V d\mathbf{r} \frac{c_v}{T^2} \left(\frac{\partial T}{\partial t} \right)^2 \leq 0.$$

with c_v the heat capacity per particle at constant volume. What do you conclude for stationary states in this example?